Suggested solutions to the IO (BSc) resit exam on August 16, 2012
VERSION: September 18, 2012

Question 1a)

It is stated in the question that a consumer buys if and only if the reservation price (or, using a synonym, the "valuation") weakly exceeds the price:

$$r \geq p.$$

Therefore, given the assumption that each consumer buys one unit or nothing ("unit demand"), the demand in the market equals the mass of all consumers who have a reservation price equal to or larger than p.

Denote the total demand in the market by Q. We can obtain an expression for Q by integrating the mass of consumers, $f(r)$, over the range of r's that weakly exceed p. Doing that yields

$$\begin{aligned} Q &= \int_p^1 f(r)\,dr = \int_p^1 mx(1-r)^{x-1}\,dr \\ &= m\left[-(1-r)^x\right]_p^1 = m\left[0 + (1-p)^x\right] \\ &= m(1-p)^x. \end{aligned}$$

This means that the *direct* demand function is given by $\boxed{Q = m(1-p)^x}$. By inverting this equation (i.e., by solving for p) we obtain the *inverse* demand function that is asked about in the question:

$$Q = m(1-p)^x \Leftrightarrow \left(\frac{Q}{m}\right)^{\frac{1}{x}} = (1-p) \Leftrightarrow p = 1 - \left(\frac{Q}{m}\right)^{\frac{1}{x}}$$

or

$$\boxed{p = 1 - bQ^y,}$$

where

$$b \equiv m^{-\frac{1}{x}}$$

and

$$y \equiv \frac{1}{x}.$$

Question 1b)

Firm i 's profit is

$$\begin{aligned} \pi_i &= pq_i - cq_i = \left[1 - c - bQ^y\right] q_i \\ &= \left[1 - c - b\left(\sum_{j=1}^n q_j\right)^y\right] q_i, \end{aligned}$$

where q_i denotes form i's quantity (and therefore $\sum_{j=1}^n q_j = Q$). The first-order condition is

$$\frac{\partial \pi_i}{\partial q_i} = \left[1 - c - b\left(\sum_{j=1}^n q_j\right)^y\right] - by\left(\sum_{j=1}^n q_j\right)^{y-1} q_i = 0. \qquad (1)$$

Since all firms are identical, we expect the equilibrium to be symmetric. Imposing symmetry in the above first-order condition yields

$$[1 - c - b(nq)^y] - by(nq)^{y-1} q = 0$$

or

$$1 - c - bn^y q^y - byn^{y-1} q^y = 0$$

or

$$1 - c = bn^{y-1} q^y (y + n)$$

or

$$q^y = \frac{1-c}{bn^{y-1}(y+n)}$$

or

$$\boxed{q^* = \left[\frac{1-c}{bn^{y-1}(y+n)}\right]^{\frac{1}{y}}.} \tag{2}$$

That is, in the (unique) symmetric equilibrium, each firm's equilibrium quantity is given by (2).

Question 1c)

The second-order condition will be satisfied if the profit function π_i is strictly concave in q_i. However, it suffices if π_i is strictly quasi-concave, which it will be if the second derivative of π_i w.r.t. q_i is strictly negative at any value of q_i that satisfies the first-order condition.

Differentiate the profit function a second time w.r.t. q_i:

$$\frac{\partial^2 \pi_i}{\partial q_i^2} = -2by\left(\sum_{j=1}^n q_j\right)^{y-1} - by(y-1)\left(\sum_{j=1}^n q_j\right)^{y-2} q_i.$$

We therefore have

$$\frac{\partial^2 \pi_i}{\partial q_i^2} < 0 \Leftrightarrow by(1-y)\left(\sum_{j=1}^n q_j\right)^{y-2} q_i < 2by\left(\sum_{j=1}^n q_j\right)^{y-1} \Leftrightarrow (1-y) q_i < 2\sum_{j=1}^n q_j. \tag{3}$$

This inequality is clearly satisfied for all $y > 1$. Therefore, the objective function is strictly concave for those values of y, and the second-order condition is fine.

For $y \leq 1$, the objective function will not necessarily be strictly concave, but it will be strictly quasi-concave for all relevant values of q_i. To see this, let a value of q_i that satisfies the first-order condition be denoted by $\widehat{q}_i$. We have

$$\frac{\partial \pi_i}{\partial q_i} = 0 \Leftrightarrow \left[1 - c - b\left(\sum_{j=1}^n q_j\right)^y\right] = by\left(\sum_{j=1}^n q_j\right)^{y-1} \widehat{q}_i \Leftrightarrow \widehat{q}_i = \frac{1 - c - b\left(\sum_{j=1}^n q_j\right)^y}{by\left(\sum_{j=1}^n q_j\right)^{y-1}}.$$

Also note that if firm i chooses $q_i = \widehat{q}_i$, then we must have $p > c$; if not, then the first-order condition could not be satisfies (see (1)). Now evaluate (3) at $\widehat{q}_i$:

$$(1-y)\widehat{q}_i < 2\sum_{j=1}^n q_j \Leftrightarrow (1-y)\left[\frac{1 - c - b\left(\sum_{j=1}^n q_j\right)^y}{by\left(\sum_{j=1}^n q_j\right)^{y-1}}\right] < 2\sum_{j=1}^n q_j \Leftrightarrow$$

$$(1-y)\left[1-c-b\left(\sum_{j=1}^{n} q_j\right)^y\right] < 2by\left(\sum_{j=1}^{n} q_j\right)^y \Leftrightarrow$$

$$(1-y)(1-c) < [2y+(1-y)]\, b\left(\sum_{j=1}^{n} q_j\right)^y \Leftrightarrow$$

$$(1-y)(1-c) < (1+y)\, b\left(\sum_{j=1}^{n} q_j\right)^y \Leftrightarrow$$

$$(1-y)(1-c) < (1+y)(1-p)\,,$$

which holds for all $y > 0$ if $p \geq c$, which we know must hold. It follows that, for $y \leq 1$, the objective function is strictly quasi-concave for all relevant values of q_i.

Conclusion: the second-order condition is satisfied for all $y > 0$.

Question 2.

- To the external examiner: This question is identical to one in the regular exam in this course from May 2009. The students had access to this exam and the answers while preparing for their own exam.

a) The game consists of two stages. At the first stage the owners choose, independently and simultaneously, an instruction P_i or R_i. At the second stage we have four different possibilities, depending on what instructions the owners have chosen: both firms are profit maximizers, (P_1, P_2); both firms are revenue maximizers, (R_1, R_2); or one is a profit maximizer and the other is a revenue maximizer, (P_1, R_2) or (R_1, P_2). Given these objectives, the managers choose, independently and simultaneously, a quantity q_i.

- We can solve for the subgame-perfect Nash equilibria of the model by backward induction. We therefore start by solving the four second-stage subgames.
- **The case** (P_1, P_2)**.** Each firm maximizes

$$\begin{aligned} & [45 - 9(q_1 + q_2)]\, q_i - 9q_i \\ = \; & [36 - 9(q_1 + q_2)]\, q_i. \end{aligned}$$

  The FOCs for the two firms are

$$-9q_1 + [36 - 9(q_1 + q_2)] = 0$$

  and

$$-9q_2 + [36 - 9(q_1 + q_2)] = 0.$$

  Solving these equations for q_1 and q_2 yields

$$\left(q_1^{PP}, q_2^{PP}\right) = \left(\frac{4}{3}, \frac{4}{3}\right).$$

  The profit levels given these outputs are

$$\pi_1^{PP} = \left[45 - 9\left(q_1^{PP} + q_2^{PP}\right)\right] q_1^{PP} - 9q_1^{PP} = 16$$

  and

$$\pi_2^{PP} = \left[45 - 9\left(q_1^{PP} + q_2^{PP}\right)\right] q_2^{PP} - 9q_2^{PP} = 16.$$

- **The case** (R_1, R_2)**.** Each firm maximizes its revenues

$$[45 - 9(q_1 + q_2)]\, q_i.$$

  The FOCs for the two firms are

$$-9q_1 + [45 - 9(q_1 + q_2)] = 0$$

  and

$$-9q_2 + [45 - 9(q_1 + q_2)] = 0.$$

Solving these equations for q_1 and q_2 yields

$$\left(q_1^{RR}, q_2^{RR}\right) = \left(\frac{5}{3}, \frac{5}{3}\right).$$

The profit levels given these outputs are

$$\pi_1^{RR} = \left[45 - 9\left(q_1^{RR} + q_2^{RR}\right)\right] q_1^{RR} - 9q_1^{RR} = 10$$

and

$$\pi_2^{RR} = \left[45 - 9\left(q_1^{RR} + q_2^{RR}\right)\right] q_2^{RR} - 9q_2^{RR} = 10.$$

- **The case** (P_1, R_2)**.** Firm 1 maximizes its profit

$$\begin{aligned} & [45 - 9(q_1 + q_2)] q_i - 9q_i \\ = \; & [36 - 9(q_1 + q_2)] q_i. \end{aligned}$$

Firm 1's FOC is

$$-9q_1 + [36 - 9(q_1 + q_2)] = 0. \tag{4}$$

Firm 2 maximizes its revenues

$$[45 - 9(q_1 + q_2)] q_i.$$

Firm 2's FOC is

$$-9q_1 + [45 - 9(q_1 + q_2)] = 0. \tag{5}$$

Solving equations (4) and (5) for q_1 and q_2 yields

$$\left(q_1^{PR}, q_2^{PR}\right) = (1, 2).$$

The profit levels given these outputs are

$$\pi_1^{PR} = \left[45 - 9\left(q_1^{PR} + q_2^{PR}\right)\right] q_1^{PR} - 9q_1^{PR} = 9$$

and

$$\pi_2^{PR} = \left[45 - 9\left(q_1^{PR} + q_2^{PR}\right)\right] q_2^{PR} - 9q_2^{PR} = 18.$$

- **The case** (R_1, P_2)**.** This is symmetric to the case (P_1, R_2). Therefore, $\left(q_1^{RP}, q_2^{RP}\right) = (2, 1)$,

$$\pi_1^{RP} = 18,$$

and

$$\pi_2^{RP} = 9.$$

- We have now solved all the stage 2 subgames and derived expressions for the equilibrium profit levels in all of these. Using these profit levels we can illustrate the stage 1 interaction between O_1 and O_2 in a game matrix (where O_1 is the row player and O_2 is the column player):

	P_2	R_2
P_1	16, 16	9, 18
R_1	18, 9	10, 10

We see that each player has a strictly dominant strategy and that, in particular, the unique Nash equilibrium of the stage 1 game is that both owners choose revenue maximization, (R_1, R_2).

- **Conclusion:** the game has a unique SPNE. In this equilibrium, both owners choose revenue maximization, (R_1, R_2). In the stage 2 equilibrium path subgame, the managers choose $\left(q_1^{RR}, q_2^{RR}\right) = \left(\frac{5}{3}, \frac{5}{3}\right)$. In the three off-the-equilibrium path subgames, the managers choose $\left(q_1^{PP}, q_2^{PP}\right) = \left(\frac{4}{3}, \frac{4}{3}\right)$, $\left(q_1^{PR}, q_2^{PR}\right) = (1, 2)$, and $\left(q_1^{RP}, q_2^{RP}\right) = (2, 1)$.

b) **Interpretation:** The owners would be better off if they both chose to instruct their manager to maximize profit. The reason why this cannot be part of an equilibrium is that each firm can gain by unilaterally instruct its own manager to maximize revenues instead. Why is this the case? First, a manager who maximizes revenues will be more aggressive (i.e., produce more) than a profit maximizing manager. Second, the rival manager, expecting this behavior, will respond by producing less (since the firms' outputs are strategic substitutes). This will increase the first firm's market share and profit.

- If the managers' choice variables had been strategic complements instead we should expect the opposite result: each firm would like to make the rival behave in a way that is good for the own profits (i.e., charge a high price or choose a small quantity). If the choice variables are strategic complements, this means that to induce the rival to behave like that a firm should behave in the same way itself (i.e., charge a high price or choose a small quantity). Therefore, an owner could gain by instructing its manager to be relatively non-aggressive (i.e., to have a strong incentive to charge a high price or choose a small quantity) — this can be achieved by instructing the manager to maximize profits rather than revenues.
- The assumption that the instruction is observable for the rival firm is crucial. Without that assumption, an owner would always want the own manager to maximize profits (but maybe still be *telling* the rival manager that the instruction was R). The point with choosing R is that then the rival *knows* this (and knows that this choice is irreversible), which will (in the model with strategic substitutes) have a beneficial effect on the rival manager's optimal choice at the second stage.